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On convergence in mixed integer programming

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Abstract Let $P \subseteq \mathbb{R}^{m+n}$ be a rational polyhedron, and let P_I be the convex hull of $P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$. We define the *integral lattice-free closure* of P as the set obtained from P by adding all inequalities obtained from disjunctions associated with integral lattice-free polyhedra in $\mathbb{R}^m$. We show that the integral lattice-free closure of P is again a polyhedron, and that repeatedly taking the integral lattice-free closure of P gives P_I after a finite number of iterations. Such results can be seen as a mixed integer analogue of theorems by Chvátal and Schrijver for the pure integer case. One ingredient of our proof is an extension of a result by Owen and Mehrotra. In fact, we prove that for each rational polyhedron P, the split closures of P yield in the limit the set P_I.

Keywords Convergence · Cutting planes · Disjunctive programming · Lattice-free polyhedra · Mixed integer programming · Split cuts

Mathematics Subject Classification (2000) 90C10 · 90C11 · 40A05

1 Introduction

Cutting plane techniques have been one of the prominent topics in the theory of integer and mixed integer programming. A fundamental result in the theory of cutting planes was shown by Chvátal [4] and Schrijver [13], and was inspired by Gomory's [6] early work. To state such result we recall the notion of Gomory cuts. An inequality $cx \leq \lfloor \gamma \rfloor$ is a *Gomory cut* for $P \subseteq \mathbb{R}^m$ if $c \in \mathbb{Z}^m$ and if $cx \leq \gamma$ is valid for P. The *Chvátal*

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closure P' of P is the set of all vectors that satisfy every Gomory cut for P. We denote by $P^{(i)}$, $i \in \mathbb{N}$, the *i-th Chvátal closure* of P, i.e. $P^{(i)} := (P^{(i-1)})'$, where $P^{(0)} := P$. We recall that a polyhedron is called *rational* if it can be described by a finite system of linear inequalities with rational data. A series of results of Chvátal and Schrijver gives the following theorem, where we denote by P_I the convex hull of $P \cap \mathbb{Z}^m$.

Theorem 1 *For each rational polyhedron P, then*

(i) *P' is again a rational polyhedron,*
(ii) *$P^{(k)} = P_I$ for some integer k.*

In a mixed integer programming problem, only some of the variables are restricted to integer values. Then the set of feasible solutions to such a problem attains the form

$$\{(x, y) \in P : x \in \mathbb{Z}^m\},$$

where P is a polyhedron in $\mathbb{R}^{m+n}$. Note that, with a slight abuse of notation we write column vectors in $\mathbb{R}^{m+n}$ in the form (x, y). The vectors $(x, y) \in \mathbb{R}^{m+n}$ such that $x \in \mathbb{Z}^m$ are called *x-integral*. We denote by P_I the convex hull of the x-integral vectors in P, and we say that P is *x-integral* if $P = P_I$. It is well-known that a rational polyhedron P is x-integral if and only if each minimal face of P contains x-integral vectors. Moreover, if P is rational then P_I is a rational polyhedron (see [9], [14, Sect. 16.7]). In the pure integer setting, i.e. when $n = 0$, we call vectors and polyhedra simply *integral*, instead of x-integral.

An inequality $cx + dy \leq \gamma$ is a *split cut* for $P \subseteq \mathbb{R}^{m+n}$ if there exists a vector $a \in \mathbb{Z}^m$ and an integer β such that $cx + dy \leq \gamma$ is valid for both

$$\{(x, y) \in P : ax \leq \beta\} \quad \text{and} \quad \{(x, y) \in P : ax \geq \beta + 1\}.$$

The *split closure* of P is defined as the set of all vectors that satisfy every split cut for P. Given $P \subseteq \mathbb{R}^{m+n}$, we denote by $\mathcal{S}(P)$ its split closure. Moreover, for every $i \in \mathbb{N}$, we denote by $\mathcal{S}^i(P)$ the *i-th split closure* of P, i.e. $\mathcal{S}^i(P) := \mathcal{S}(\mathcal{S}^{i-1}(P))$, where $\mathcal{S}^0(P) := P$. Cook et al. [5] proved that the split closure of a rational polyhedron $P \subseteq \mathbb{R}^{m+n}$ is again a rational polyhedron. In the general mixed integer case, Cook et al. showed that determining split closures does not suffice to generate P_I in a finite number of iterations. The reason is that, even if $\mathcal{S}^i(P) \neq P_I$ implies that $\mathcal{S}^{i+1}(P) \subset \mathcal{S}^i(P)$, the difference between $\mathcal{S}^i(P)$ and $\mathcal{S}^{i+1}(P)$ may become arbitrarily small as i grows.

Cook et al. also showed that, given a rational polyhedron P, P_I can be generated in a finite number of iterations by combining split closures with certain rounding cuts that correspond to a fixed discretization of the continuous variables based on the original system defining P. However, it remained a challenge for many years to design finite cutting plane algorithms that directly work in the original mixed integer setting, without discretizing the continuous variables, and without remembering the original system.

Split closures alone do not lead to finite convergence to P_I even if P is bounded. In this special case, however, Owen and Mehrotra [10] showed that the sequence of split closures does yield in the limit the set P_I. In Sect. 2 we extend the result of Owen and Mehrotra from polytopes to rational polyhedra. That is, we prove that for each rational polyhedron P, the repeated computation of the split closures yields in the limit the set P_I. This result is a backbone of the main theorem of our paper, which can be seen as an analogue of Theorem 1 in mixed integer programming. In order to state it precisely, we next introduce the notion of lattice-free polyhedra.

A polyhedron $L \subseteq \mathbb{R}^m$ is said to be *lattice-free* if $\operatorname{relint}(L) \cap \mathbb{Z}^m = \emptyset$. (We recall that relint L denotes the *relative interior* of a polyhedron L, which is the set of points x for which there exists a ball centered in x whose intersection with the affine hull of L is contained in L.) An inequality $cx+dy \leq \gamma$ is an *integral lattice-free cut* for $P \subseteq \mathbb{R}^{m+n}$ if there exists an integral lattice-free polyhedron $\{x \in \mathbb{R}^m : a_j x \leq \beta_j,\ j \in J\}$ such that $cx + dy \leq \gamma$ is valid for every set

$$\{(x, y) \in P : a_j x \geq \beta_j\}, \quad j \in J.$$

It is easy to see that an integral lattice-free cut is satisfied by all x-integral vectors in P. Clearly, every split cut for P is also an integral lattice-free cut for P. The *integral lattice-free closure* of P is defined as the set of all vectors that satisfy every integral lattice-free cut for P. Given $P \subseteq \mathbb{R}^{m+n}$, we denote by $\mathcal{L}(P)$ its integral lattice-free closure. Moreover, for every $i \in \mathbb{N}$, we denote by $\mathcal{L}^i(P)$ the *i-th integral lattice-free closure* of P, i.e. $\mathcal{L}^i(P) := \mathcal{L}(\mathcal{L}^{i-1}(P))$, where $\mathcal{L}^0(P) := P$.

We are now ready to state the main result of our paper. In Sect. 3 we prove that, if P is a rational polyhedron, then repeatedly taking the integral lattice-free closure of P gives P_I after a finite number of iterations. Moreover, we prove that the integral lattice-free closure of a rational polyhedron P is again a rational polyhedron. We also show that in general it is not true that the integral lattice-free closure of P equals P_I.

From now on in this paper, if not explicitly stated, we work with rational spaces, rather than real ones. In particular, any matrix, any vector, and any polyhedron is supposed to be rational. Moreover, given sets P and Q, we define $P + Q := \{p + q : p \in P,\ q \in Q\}$. Finally, we denote by $\mathbb{N}$ the set of nonnegative integers, and by B be the unit ball.

2 Infinite convergence

Let $\{\tilde{P}, P^i : i \in \mathbb{N}\}$ be a family of closed sets such that $\tilde{P} \subseteq P^{i+1} \subseteq P^i$ for every $i \in \mathbb{N}$. We say that the sequence $\{P^i : i \in \mathbb{N}\}$ *(Hausdorff) converges* to $\tilde{P}$, and we write $\lim_{i\to\infty} P^i = \tilde{P}$, if for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $P^k \subseteq \tilde{P} + \epsilon B$. The given definition of convergence is based on the well-known Hausdorff distance, see [12, Sect. 3] for more details. Note that, if $\tilde{P} = \emptyset$, then it follows by definition that there exists $k \in \mathbb{N}$ such that $P^i = \emptyset$ for all $i \geq k$. It is a well-known fact that if the sequence $\{P^i : i \in \mathbb{N}\}$ converges, then $\lim_{i\to\infty} P^i = \bigcap_{i\in\mathbb{N}} P^i$ (see for example [12, Theorem 2, Proposition 2]).

The main result of this section is that $\lim_{i\to\infty} \mathcal{S}^i(P) = P_I$, for each polyhedron P. In the remainder of this section, we develop the proof of such statement. Note that in the special case when P is a polytope, this result has been shown by Owen and Mehrotra [10]. We apply their overall proof strategy. However, several technical results are necessary to provide the proof for general polyhedra. We now give some easy observations about Hausdorff convergence that we need later, only sketching their proofs.

Given a set of vectors P, in what follows we denote with conv.hull P, lin.hull P, aff.hull P respectively the convex hull, the linear hull, and the affine hull of P. We refer to [14] for the standard definitions of such concepts. We also denote with cl P the topological closure of P.

Observation 1 *Let* $\{\tilde{P}, P^i, Q^i \ : \ i \in \mathbb{N}\}$ *be a family of closed sets with* $\tilde{P} \subseteq P^{i+1} \subseteq P^i$, $\tilde{P} \subseteq Q^{i+1} \subseteq Q^i$, $Q^i \subseteq P^i$ *for every* $i \in \mathbb{N}$, $\lim_{i\to\infty} P^i = \tilde{P}$. *Then*

$$\lim_{i\to\infty} Q^i = \tilde{P}.$$

Proof Let $\epsilon > 0$. Since $\lim_{i\to\infty} P^i = \tilde{P}$, there exists $k \in \mathbb{N}$ such that $P^k \subseteq \tilde{P} + \epsilon B$. This implies that $Q^k \subseteq P^k \subseteq \tilde{P} + \epsilon B$. □

Observation 2 *Let* $\{\tilde{P}, \tilde{Q}, P^i, Q^i \ : \ i \in \mathbb{N}\}$ *be a family of closed sets with* $\tilde{P} \subseteq P^{i+1} \subseteq P^i$, $\tilde{Q} \subseteq Q^{i+1} \subseteq Q^i$ *for every* $i \in \mathbb{N}$, $\lim_{i\to\infty} P^i = \tilde{P}$, $\lim_{i\to\infty} Q^i = \tilde{Q}$. *Then*

$$\lim_{i\to\infty} (P^i \cup Q^i) = \tilde{P} \cup \tilde{Q}.$$

Proof Clearly $\{\tilde{P} \cup \tilde{Q}, P^i \cup Q^i \ : \ i \in \mathbb{N}\}$ is a family of closed sets with $\tilde{P} \cup \tilde{Q} \subseteq P^{i+1} \cup Q^{i+1} \subseteq P^i \cup Q^i$ for every $i \in \mathbb{N}$.

Let $\epsilon > 0$. Since $\lim_{i\to\infty} P^i = \tilde{P}$, and $\lim_{i\to\infty} Q^i = \tilde{Q}$, there exists $k \in \mathbb{N}$ such that $P^k \subseteq \tilde{P} + \epsilon B$, and $Q^k \subseteq \tilde{Q} + \epsilon B$. This implies that $P^k \cup Q^k \subseteq (\tilde{P} + \epsilon B) \cup (\tilde{Q} + \epsilon B) = (\tilde{P} \cup \tilde{Q}) + \epsilon B$. □

Observation 3 *Let* $\{\tilde{P}, P^i \ : \ i \in \mathbb{N}\}$ *be a family of closed sets with* $\tilde{P} \subseteq P^{i+1} \subseteq P^i$ *for every* $i \in \mathbb{N}$, $\lim_{i\to\infty} P^i = \tilde{P}$. *Then*

$$\lim_{i\to\infty} \operatorname{cl} \operatorname{conv.hull} P^i = \operatorname{cl} \operatorname{conv.hull} \tilde{P}.$$

Proof Clearly $\{\operatorname{cl} \operatorname{conv.hull} \tilde{P}, \operatorname{cl} \operatorname{conv.hull} P^i \ : \ i \in \mathbb{N}\}$ is a family of closed sets with $\operatorname{cl} \operatorname{conv.hull} \tilde{P} \subseteq \operatorname{cl} \operatorname{conv.hull} P^{i+1} \subseteq \operatorname{cl} \operatorname{conv.hull} P^i$ for every $i \in \mathbb{N}$.

Let $\epsilon > 0$. Since $\lim_{i\to\infty} P^i = \tilde{P}$, there exists $k \in \mathbb{N}$ such that $P^k \subseteq \tilde{P} + \epsilon B$. This implies that $\operatorname{cl} \operatorname{conv.hull} P^k \subseteq \operatorname{cl} \operatorname{conv.hull}(\tilde{P} + \epsilon B)$. Since ϵB is convex, it is easy to verify that $\operatorname{conv.hull}(\tilde{P} + \epsilon B) = (\operatorname{conv.hull} \tilde{P}) + \epsilon B$. Since moreover ϵB is compact, it can be checked that $\operatorname{cl} \operatorname{conv.hull}(\tilde{P} + \epsilon B) = \operatorname{cl}((\operatorname{conv.hull} \tilde{P}) + \epsilon B) = (\operatorname{cl} \operatorname{conv.hull} \tilde{P}) + \epsilon B$. Hence $\operatorname{cl} \operatorname{conv.hull} P^k \subseteq (\operatorname{cl} \operatorname{conv.hull} \tilde{P}) + \epsilon B$. □

Observation 4 *Let* $\{\tilde{P}, \tilde{Q}, P^i, Q^i \ : \ i \in \mathbb{N}\}$ *be a family of closed sets with* $\tilde{P} \subseteq P^{i+1} \subseteq P^i$, $\tilde{Q} \subseteq Q^{i+1} \subseteq Q^i$, $P^i \subseteq Q^i$ *for every* $i \in \mathbb{N}$, $\lim_{i\to\infty} P^i = \tilde{P}$, $\lim_{i\to\infty} Q^i = \tilde{Q}$. *Then*

$$\tilde{P} \subseteq \tilde{Q}.$$

Proof Since $P^i \subseteq Q^i$ for every $i \in \mathbb{N}$, it follows that $\bigcap_{i\in\mathbb{N}} P^i \subseteq \bigcap_{i\in\mathbb{N}} Q^i$. The statement follows since $\tilde{P} = \bigcap_{i\in\mathbb{N}} P^i$, and $\tilde{Q} = \bigcap_{i\in\mathbb{N}} Q^i$. □

Observation 5 *Let $\{\tilde{P}, P^i : i \in \mathbb{N}\}$ be a family of closed sets with $\tilde{P} \subseteq P^{i+1} \subseteq P^i$ for every $i \in \mathbb{N}$, where $\tilde{P}$ is the polyhedron defined by the finite system $c_j z \leq \gamma_j$, $j \in J$. Then $\lim_{i\to\infty} P^i = \tilde{P}$ if and only if for every $j \in J, \epsilon > 0$, there exists $k \in \mathbb{N}$ such that $c_j z \leq \gamma_j + \epsilon$ is valid for P^k.*

Proof At first we prove necessity of the condition. By hypothesis, for every $\delta > 0$, there exists $k \in \mathbb{N}$ such that $P^k \subseteq \tilde{P} + \delta B$. Let $j \in J$. Since $\tilde{P}$ is a polyhedron, it can be seen that there exists $\lambda > 0$, independent on δ, such that $c_j z \leq \gamma_j + \lambda\delta$ is valid for P^k. Now let $\epsilon > 0$. The statement follows by choosing δ such that $\lambda\delta \leq \epsilon$.

We now prove sufficiency of the condition. Since J is finite, it follows by hypothesis that for every $\delta > 0$, there exists $k \in \mathbb{N}$ such that $c_j z \leq \gamma_j + \delta$ is valid for P^k for every $j \in J$. If $\tilde{P} = \emptyset$, there exists $\delta > 0$ such that the system $c_j z \leq \gamma_j + \delta$, $j \in J$, is infeasible, and so in this case $P^k = \emptyset$, and we are done. Otherwise, since $\tilde{P}$ is a polyhedron, it can be seen that there exists $\lambda > 0$, independent on δ, such that $P^k \subseteq \tilde{P} + \lambda\delta B$. Let $\epsilon > 0$. The statement follows by choosing δ such that $\lambda\delta \leq \epsilon$. □

Observation 6 *Let $\{\tilde{P}, \tilde{Q}, P^i, Q^i : i \in \mathbb{N}\}$ be a family of closed sets with $\tilde{P} \subseteq P^{i+1} \subseteq P^i$, $\tilde{Q} \subseteq Q^{i+1} \subseteq Q^i$ for every $i \in \mathbb{N}$, $\lim_{i\to\infty} P^i = \tilde{P}$, $\lim_{i\to\infty} Q^i = \tilde{Q}$, and where $\tilde{P}$, $\tilde{Q}$ are polyhedra. Then*

$$\lim_{i\to\infty} (P^i \cap Q^i) = \tilde{P} \cap \tilde{Q}.$$

Proof Clearly $\{\tilde{P} \cap \tilde{Q}, P^i \cap Q^i : i \in \mathbb{N}\}$ is a family of closed sets with $\tilde{P} \cap \tilde{Q} \subseteq P^{i+1} \cap Q^{i+1} \subseteq P^i \cap Q^i$ for every $i \in \mathbb{N}$.

Let $c^P_j z \leq \gamma^P_j$, $j \in J^P$, be a finite system defining $\tilde{P}$, and let $c^Q_j z \leq \gamma^Q_j$, $j \in J^Q$, be a finite system defining $\tilde{Q}$. Since $\lim_{i\to\infty} P^i = \tilde{P}$, $\lim_{i\to\infty} Q^i = \tilde{Q}$, it follows by Observation 5 that for every $j \in J^P, \epsilon > 0$, there exists $k \in \mathbb{N}$ such that $c^P_j z \leq \gamma^P_j + \epsilon$ is valid for P^k, and that for every $j \in J^Q, \epsilon > 0$, there exists $k \in \mathbb{N}$ such that $c^Q_j z \leq \gamma^Q_j + \epsilon$ is valid for Q^k. Since the polyhedron $\tilde{P} \cap \tilde{Q}$ is defined by the finite system $c^P_j z \leq \gamma^P_j$, $j \in J^P$, $c^Q_j z \leq \gamma^Q_j$, $j \in J^Q$, it follows by Observation 5 that $\lim_{i\to\infty}(P^i \cap Q^i) = \tilde{P} \cap \tilde{Q}$. □

One important ingredient of our proof is the following result about existence of a hyperplane which preserves mixed integrality under projection along a vector. To make this precise, we introduce the following notation. Given a nonzero vector $v \in \mathbb{Q}^{m+n}$, a subspace H of $\mathbb{R}^{m+n}$ of dimension $m + n - 1$ such that $v \notin H$, and a set W in $\mathbb{R}^{m+n}$, we denote by $\operatorname{proj}_{v,H} W$ the projection of W to the subspace H along the direction v, i.e. $\operatorname{proj}_{v,H} W = \{z \in H : \exists\lambda \in \mathbb{R}, z + \lambda v \in W\}$. Moreover we denote with $\operatorname{proj}_x W$ the orthogonal projection of W onto the space of the x-variables, i.e. $\operatorname{proj}_x W = \{x \in \mathbb{R}^m : \exists y \in \mathbb{R}^n, (x, y) \in W\}$. Given a nonzero vector $v \in \mathbb{Q}^{m+n}$, and a subspace H of $\mathbb{R}^{m+n}$ of dimension $m + n - 1$, we say that H is *mixed integer* *invariant under projection along* v, if $v \notin H$, and if $\operatorname{proj}_{v,H} w \in \mathbb{Z}^m \times \mathbb{R}^n$ for every vector $w \in \mathbb{Z}^m \times \mathbb{R}^n$.

Lemma 7 *(Mixed integer invariance under projection) Let* v *be a nonzero vector in* $\mathbb{Q}^{m+n}$. *Then there exists a subspace* H *of* $\mathbb{R}^{m+n}$ *that is mixed integer invariant under projection along* v.

Proof Let $v_x := \operatorname{proj}_x v$. If $v_x = 0$, then the result follows trivially by taking any subspace H of $\mathbb{R}^{m+n}$ of dimension $m + n - 1$ such that $v \notin H$. So we now assume $v_x \neq 0$. By scaling, we can assume that v_x is integral with $\gcd(v_1, \ldots, v_m) = 1$. Then it is well-known (see for example [7, Theorem 5 on page 21]) that there exists a lattice basis V of $\mathbb{Z}^m$ containing v_x. Let $\overline{H} = \{x \in \mathbb{R}^m : hx = 0\}$ be the subspace of dimension $m - 1$ of $\mathbb{R}^m$ spanned by the vectors in $V \setminus \{v_x\}$. Clearly $v_x \notin \overline{H}$, and since V is a basis of $\mathbb{Z}^m$, it follows that $\operatorname{proj}_{v_x,\overline{H}} \overline{w} \in \mathbb{Z}^m$ for every vector $\overline{w} \in \mathbb{Z}^m$.

Now let $H = \{(x, y) \in \mathbb{R}^{m+n} : hx = 0\}$. Clearly, the dimension of H is $m + n - 1$. Since $v_x \notin \overline{H}$, then $hv_x \neq 0$, hence $v \notin H$. Let $w \in \mathbb{Z}^m \times \mathbb{R}^n$, and let $w_x := \operatorname{proj}_x w$. We now show that $\operatorname{proj}_x(\operatorname{proj}_{v,H} w) = \operatorname{proj}_{v_x,\overline{H}}(w_x)$. Notice that $\operatorname{proj}_{v,H} w = w + \lambda v$, where λ is a scalar with $h(w_x + \lambda v_x) = 0$. This implies that $\operatorname{proj}_x(\operatorname{proj}_{v,H} w) = w_x + \lambda v_x$. On the other hand, $\operatorname{proj}_{v_x,\overline{H}}(w_x) = w_x + \mu v_x$ for a scalar μ with $h(w_x + \mu v_x) = 0$. Hence $\operatorname{proj}_x(\operatorname{proj}_{v,H} w) = \operatorname{proj}_{v_x,\overline{H}}(w_x)$. Since $w_x \in \mathbb{Z}^m$, it follows from the first part of the proof that $\operatorname{proj}_{v_x,\overline{H}}(w_x) \in \mathbb{Z}^m$, thus $\operatorname{proj}_x(\operatorname{proj}_{v,H} w) \in \mathbb{Z}^m$, which completes the proof. $\square$

The next two lemmas establish important properties of the subspace H that is mixed integer invariant under projection along v. Their proofs are given in Sect. 4. We recall that the characteristic cone of a set P is $\operatorname{char.cone} P := \{w : z+w \in P \text{ for all } z \in P\}$, and the lineality space of P is $\operatorname{lin.space} P := (\operatorname{char.cone} P) \cap (-\operatorname{char.cone} P)$. A set P is called *pointed* if $\operatorname{lin.space} P$ has dimension zero.

Lemma 8 *Let* P *be a polyhedron in* $\mathbb{R}^{m+n}$, *let* $v \in \mathbb{Q}^{m+n}$ *be a nonzero vector, and let* H *be a subspace that is mixed integer invariant under projection along* v. *Then*

$$\operatorname*{proj}_{v,H} \mathcal{S}^i(P) \subseteq \mathcal{S}^i(\operatorname*{proj}_{v,H} P) \quad \textit{for every } i \in \mathbb{N}.$$

Lemma 9 *Let* P *be an unbounded polyhedron in* $\mathbb{R}^{m+n}$, *let* v *be a nonzero vector in* char.cone P, *and let* H *be a subspace that is mixed integer invariant under projection along* v. *Then*

$$\operatorname*{proj}_{v,H}(P_I) = (\operatorname*{proj}_{v,H} P)_I.$$

We are now prepared to prove the main result of this section.

Theorem 2 *For each rational polyhedron* P, $\lim_{i\to\infty} \mathcal{S}^i(P) = P_I$.

Proof Let $P \subseteq \mathbb{R}^{m+n}$ be a polyhedron, and notice that $P_I \subseteq \mathcal{S}^{i+1}(P) \subseteq \mathcal{S}^i(P)$ for every $i \in \mathbb{N}$. Moreover by Cook et al. [5], $\{P_I, \mathcal{S}^i(P) : i \in \mathbb{N}\}$ is a family of polyhedra, thus of closed sets. The proof is by induction on the dimension of P, the base cases $\dim P = -1$ (i.e. $P = \emptyset$) and $\dim P = 0$ (i.e. P is a singleton) being trivial. To prove the inductive step, let $\dim P \geq 1$, and assume that the statement is true for every polyhedron of dimension strictly smaller than $\dim P$. If P is bounded, then the result follows from Owen and Mehrotra [10]. Thus, from now on, we assume that P is unbounded.

In the remainder of the proof, we use the following fact (that follows from [14, Theorem 16.1]). If P_I is nonempty, then

$$\text{char.cone}(P_I) = \text{char.cone}\, \mathcal{S}^i(P) = \text{char.cone}\, P \quad \text{for every } i \in \mathbb{N}.$$

Claim 1 *If* $\text{lin.space}\, P \neq \{0\}$, *then* $\lim_{i\to\infty} \mathcal{S}^i(P) = P_I$.

Proof of Claim Let v be a nonzero vector in $\text{lin.space}\, P$. By Lemma 7 there exists a subspace H of $\mathbb{R}^{m+n}$ that is mixed integer invariant under projection along v. Let $\overline{P} = \text{proj}_{v,H}\, P = P \cap H$. Since v is a nonzero vector in $\text{lin.space}\, P$, then $\dim \overline{P} < \dim P$. Thus by the hypothesis of the induction the sequence $\{\mathcal{S}^i(\overline{P}) \,:\, i \in \mathbb{N}\}$ converges to $\overline{P}_I$. Clearly the sequence $\{\mathcal{S}^i(\overline{P}) + \text{lin.hull}\{v\} \,:\, i \in \mathbb{N}\}$ converges to $\overline{P}_I + \text{lin.hull}\{v\}$. By Lemma 9, $\text{proj}_{v,H}(P_I) = \overline{P}_I$, and, since $v \in \text{lin.space}(P_I)$ if $P_I \neq \emptyset$, it follows that $P_I = \overline{P}_I + \text{lin.hull}\{v\}$. Hence the sequence $\{\mathcal{S}^i(\overline{P}) + \text{lin.hull}\{v\} \,:\, i \in \mathbb{N}\}$ converges to P_I. By Lemma 8, $\text{proj}_{v,H}\, \mathcal{S}^i(P) \subseteq \mathcal{S}^i(\overline{P})$ for every $i \in \mathbb{N}$, which implies that $\mathcal{S}^i(P) \subseteq \mathcal{S}^i(\overline{P}) + \text{lin.hull}\{v\}$ for every $i \in \mathbb{N}$. Since moreover $P_I \subseteq \mathcal{S}^i(P)$ for every $i \in \mathbb{N}$, it follows by Observation 1 that $\lim_{i\to\infty} \mathcal{S}^i(P) = P_I$. □

Thus, from now on, we can assume that P is unbounded and pointed.

Claim 2 *If* $P_I = \emptyset$, *then* $\lim_{i\to\infty} \mathcal{S}^i(P) = \emptyset$.

Proof of Claim Let $v \in \text{char.cone}\, P$, and let $P' = P + \text{lin.hull}\{v\}$. Clearly $P \subseteq P'$, $\dim P' = \dim P$ and $\text{lin.space}\, P' \neq \emptyset$. Moreover it is straightforward to verify that $P'_I = P_I = \emptyset$. By Claim 1, $\lim_{i\to\infty} \mathcal{S}^i(P') = \emptyset$. Since $P \subseteq P'$, then $\mathcal{S}^i(P) \subseteq \mathcal{S}^i(P')$ for every $i \in \mathbb{N}$. Thus by Observation 1, $\lim_{i\to\infty} \mathcal{S}^i(P) = \emptyset$. □

From now on we can assume that P is unbounded, pointed, and such that $P_I \neq \emptyset$. Notice that this implies that $\mathcal{S}^i(P) \neq \emptyset$ for all $i \in \mathbb{N}$.

Claim 3 *There exists a closed convex set* $\tilde{P}$, *with* $\tilde{P} \supseteq P_I$ *and* $\text{char.cone}\, \tilde{P} = \text{char.cone}\, P_I$, *such that* $\lim_{i\to\infty} \mathcal{S}^i(P) = \tilde{P}$. *Moreover for any half-space* Q, $\lim_{i\to\infty} (\mathcal{S}^i(P) \cap Q) = \tilde{P} \cap Q$.

Proof of Claim Since P is pointed and $P_I \neq \emptyset$, it follows that also P_I is pointed. Hence it is easy to see that there exists a half-space

$$R^{\geq} = \{z \in \mathbb{R}^{m+n} \,:\, pz \geq \rho\}$$

that contains every vertex of P_I, and such that $P \cap R^{\geq}$ is bounded. Since $\{\mathcal{S}^i(P) \,:\, i \in \mathbb{N}\}$ is a sequence of polyhedra, it follows that $\{\mathcal{S}^i(P) \cap R^{\geq} \,:\, i \in \mathbb{N}\}$ is a sequence of compact sets such that $\mathcal{S}^{i+1}(P) \cap R^{\geq} \subseteq \mathcal{S}^i(P) \cap R^{\geq}$ for every $i \in \mathbb{N}$. Hence it follows from the Blaschke selection theorem [3] that $\{\mathcal{S}^i(P) \cap R^{\geq} \,:\, i \in \mathbb{N}\}$ converges to a set $\hat{P}$. It follows that $\hat{P} := \bigcap_{i\in\mathbb{N}}(\mathcal{S}^i(P) \cap R^{\geq})$, hence $\hat{P}$ is convex and compact as it is the intersection of convex compact sets.

Similarly also the sequence of compact sets $\{\mathcal{S}^i(P) \cap R^{\geq} \cap Q \ : \ i \in \mathbb{N}\}$ converges, and it converges to the set $\bigcap_{i\in\mathbb{N}}(\mathcal{S}^i(P) \cap R^{\geq} \cap Q) = \hat{P} \cap Q$.

Now let

$$R_1 := \{z \in \mathbb{R}^{m+n} \ : \ pz \leq \rho\}.$$

We now show that $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap R_1) = P_I \cap R_1$. Notice that, since all the vertices of P_I are contained in R_2, an irredundant inequality description of $P_I \cap R_1$ is given by the inequality $pz \leq \rho$, by a system of inequalities defining the affine hull of P_I, and by the inequalities defining the unbounded facets of P_I. Note that, for each inequality of such system different from $pz \leq \rho$, the corresponding supporting hyperplane contains an unbounded face of P_I. By Observation 5, to prove that $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap R_1) = P_I \cap R_1$, we only need to prove that for every inequality $cz \leq \gamma$ of such irredundant system defining $P_I \cap R_1$, and for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $cz \leq \gamma + \epsilon$ is valid for $\mathcal{S}^k(P) \cap R_1$.

By definition of R_1, such property is trivially valid for the constraint $pz \leq \rho$, hence we can assume that $cz \leq \gamma$ is valid for P_I, and $\{z \in P_I \ : \ cz = \gamma\}$ is unbounded. Since $\mathcal{S}^i(P) \cap R_1 \subseteq \mathcal{S}^i(P)$ for every $i \in \mathbb{N}$, we only need to verify that, for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $cz \leq \gamma + \epsilon$ is valid for $\mathcal{S}^k(P)$. Now let $\epsilon > 0$, and let v be a nonzero vector in char.cone$\{z \in P_I \ : \ cz = \gamma\}$. Clearly $v \in$ char.cone(P). By Lemma 7 there exists a subspace H of $\mathbb{R}^{m+n}$ that is mixed integer invariant under projection along v. Let $\overline{P} = \text{proj}_{v,H} P$. Since v is a nonzero vector in char.cone(P), then $\dim(\overline{P}) < \dim(P)$. Thus by induction the sequence $\{\mathcal{S}^i(\overline{P}) \ : \ i \in \mathbb{N}\}$ converges to $\overline{P}_I$. Since $cz \leq \gamma$ is valid for P_I, and since $cv = 0$, it follows that $cz \leq \gamma$ is also valid for $\text{proj}_{v,H}(P_I)$. By Lemma 9, $\text{proj}_{v,H}(P_I) = \overline{P}_I$, hence $cz \leq \gamma$ is valid for $\overline{P}_I$. Since $\{\mathcal{S}^i(\overline{P}) \ : \ i \in \mathbb{N}\}$ converges to $\overline{P}_I$, there exists $k \in \mathbb{N}$ such that $cz \leq \gamma + \epsilon$ is valid for $\mathcal{S}^k(\overline{P})$. By Lemma 8, $\text{proj}_{v,H} \mathcal{S}^k(P) \subseteq \mathcal{S}^k(\overline{P})$, thus $cz \leq \gamma + \epsilon$ is valid for $\text{proj}_{v,H} \mathcal{S}^k(P)$. Finally, since $cv = 0$, it follows that $cz \leq \gamma + \epsilon$ is valid for $\mathcal{S}^k(P)$. Hence we showed that $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap R_1) = P_I \cap R_1$.

It follows from Observation 6 that the sequence of polyhedra $\{\mathcal{S}^i(P) \cap R_1 \cap Q \ : \ i \in \mathbb{N}\}$ converges to the polyhedron $P_I \cap R_1 \cap Q$.

Since $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap R_2) = \hat{P}$ and $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap R_1) = P_I \cap R_1$, then by Observation 2, $\lim_{i\to\infty} \mathcal{S}^i(P) = \hat{P} \cup (P_I \cap R_1) =: \tilde{P}$. It follows that $\tilde{P} = \bigcap_{i\in\mathbb{N}} \mathcal{S}^i(P)$. Since $\tilde{P}$ is the intersection of closed convex sets, it is itself closed and convex. Since moreover $P_I \subseteq \mathcal{S}^i(P)$ for every $i \in \mathbb{N}$, it follows that $P_I \subseteq \tilde{P}$. In particular it follows that $\tilde{P} = \hat{P} \cup P_I$, where $\hat{P}$ is convex and compact. This implies that char.cone $\tilde{P} \subseteq$ char.cone P_I. Moreover, since $\tilde{P}$ is convex and closed, it follows easily that char.cone $\tilde{P} \supseteq$ char.cone P_I.

Moreover, since $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap R_2 \cap Q) = \hat{P} \cap Q$ and $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap R_1 \cap Q) = P_I \cap R_1 \cap Q$, then by Observation 2, $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap Q) = (\hat{P} \cap Q) \cup (P_I \cap R_1 \cap Q) = (\hat{P} \cup (P_I \cap R_1)) \cap Q = \tilde{P} \cap Q$. □

Claim 4 $\mathcal{S}(\tilde{P}) = \tilde{P}$.

Proof of Claim Clearly $\mathcal{S}(\tilde{P}) \subseteq \tilde{P}$, hence now we prove the opposite inclusion. Let $a \in \mathbb{Z}^m$, $\beta \in \mathbb{Z}$, and let $Q_1 := \{(x, y) \in \mathbb{R}^{m+n} \ : \ ax \leq \beta\}$, $Q_2 := \{(x, y) \in \mathbb{R}^{m+n} \ : \ ax \geq \beta + 1\}$. Then, by definition of split closure,

$$\mathcal{S}^{i+1}(P) \subseteq \text{cl conv.hull}((\mathcal{S}^i(P) \cap Q_1) \cup (\mathcal{S}^i(P) \cap Q_2)).$$

By Claim 3, $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap Q_1) = \tilde{P} \cap Q_1$, and $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap Q_2) = \tilde{P} \cap Q_2$. By Observation 2, $\lim_{i\to\infty}((\mathcal{S}^i(P) \cap Q_1) \cup (\mathcal{S}^i(P) \cap Q_2)) = (\tilde{P} \cap Q_1) \cup (\tilde{P} \cap Q_2)$. By Observation 3, $\lim_{i\to\infty} \text{cl conv.hull}((\mathcal{S}^i(P) \cap Q_1) \cup (\mathcal{S}^i(P) \cap Q_2)) = \text{cl conv.hull}((\tilde{P} \cap Q_1) \cup (\tilde{P} \cap Q_2))$. Since moreover by Claim 3, $\lim_{i\to\infty} \mathcal{S}^{i+1}(P) = \tilde{P}$, it follows by Observation 4 that

$$\tilde{P} \subseteq \text{cl conv.hull}((\tilde{P} \cap Q_1) \cup (\tilde{P} \cap Q_2)).$$

Since this holds for every $a \in \mathbb{Z}^m$, $\beta \in \mathbb{Z}$, we have that

$$\tilde{P} \subseteq \bigcap_{(a,\beta)\in\mathbb{Z}^{m+1}} \text{cl conv.hull}(\{(x, y) \in \tilde{P} : ax \leq \beta\} \cup \{(x, y) \in \tilde{P} : ax \geq \beta + 1\}),$$

which means that $\tilde{P} \subseteq \mathcal{S}(\tilde{P})$. □

We now show that

$$\tilde{P} = P_I.$$

We prove this by contradiction, thus assume $\tilde{P} \neq P_I$. By Claim 3, $\tilde{P} \supseteq P_I$, thus $\tilde{P} \supset P_I$. Moreover $\tilde{P}$ is closed and convex. Using Rockafellar's notation [11, Sect. 18], the zero-dimensional faces of $\tilde{P}$ are called the *extreme points* of $\tilde{P}$. Moreover, an *exposed point* of $\tilde{P}$ is a point of $\tilde{P}$ through which there is a supporting hyperplane which contains no other point of $\tilde{P}$.

We now show that there exists an exposed point $\tilde{z}$ of $\tilde{P}$ with $\tilde{z} \notin P_I$. By Claim 3, we have that char.cone $\tilde{P}$ = char.cone P_I = char.cone P. Since moreover lin.space $P = \{0\}$, it follows that lin.space $\tilde{P} = \{0\}$. Since $\tilde{P}$ is closed and convex, lin.space $\tilde{P} = \{0\}$, char.cone $\tilde{P}$ = char.cone P_I, and $P_I \subset \tilde{P}$, it follows that there exists an extreme point $\bar{z}$ of $\tilde{P}$ not in P_I (see [11, Theorem 18.5]). It follows from Straszewicz's Theorem [11, Theorem 18.6] that $\bar{z}$ is the limit of some sequence of exposed points of $\tilde{P}$. Hence there exists an exposed point $\tilde{z}$ of $\tilde{P}$ arbitrarily close to $\bar{z}$. On the other hand, since P_I is a polyhedron that does not contain $\bar{z}$, there cannot be points in P_I arbitrarily close to $\bar{z}$, thus there exists an exposed point $\tilde{z}$ of $\tilde{P}$ not in P_I.

Since $\tilde{z}$ is exposed, let $cz \leq \gamma$ be an inequality valid for $\tilde{P}$, and such that $\tilde{z}$ is the only vector in $\tilde{P}$ contained in the corresponding hyperplane. Since $\tilde{z} = (\tilde{x}, \tilde{y}) \in \tilde{P} \setminus P_I$, it follows that $\tilde{x}$ is not integral, thus it contains a component not integer, say $\tilde{x}_j$. Let $Q_1 := \{(x, y) \in \mathbb{R}^{m+n} : x_j \leq \lfloor \tilde{x}_j \rfloor\}$, $Q_2 := \{(x, y) \in \mathbb{R}^{m+n} : x_j \geq \lceil \tilde{x}_j \rceil\}$. Since $\tilde{P}$ is closed and convex, so are $\tilde{P} \cap Q_1$ and $\tilde{P} \cap Q_2$. Since $\tilde{z}$ is the only point in $\tilde{P}$ that satisfies $cz = \gamma$, and $\tilde{z} \notin Q_1 \cup Q_2$, it follows that there exists $\epsilon > 0$ such that $cz \leq \gamma - \epsilon$ is valid for $\tilde{P} \cap Q_1$ and for $\tilde{P} \cap Q_2$. Hence $cz \leq \gamma - \epsilon$ is a split cut for $\tilde{P}$ which is not valid for $\tilde{z}$. It follows that $\mathcal{S}(\tilde{P}) \subset \tilde{P}$, which contradicts Claim 4. □

3 Finite convergence

In this section we prove an analogue of Theorem 1 in mixed integer programming. Indeed, we show that the integral lattice-free closure of a polyhedron is again a polyhedron, and that repeatedly taking the integral lattice-free closure of P gives P_I after a finite number of iterations. Moreover, we show that in general it is not true that the lattice-free closure of P equals P_I.

Given a full-dimensional polyhedron $L \in \mathbb{R}^m$ that contains integral points in every facet, the *max-facet-width* $w(L)$ of L is the maximum value of $\beta - \min\{ax : x \in L\}$, where $ax \leq \beta$ defines a facet of L with $a \in \mathbb{Z}^m$, and the greatest common divisor of the entries in a is 1. A result by Andersen et al. [1] shows that the closure of a polyhedron obtained from disjunctions associated with any family of maximal lattice-free polyhedra with bounded max-facet width is a rational polyhedron. In such proof, the maximality assumption is only used to show that every facet of every lattice-free polyhedron in the family contains an integral point. Since such condition is clearly satisfied by any integral polyhedron, the result by Andersen et al. implies that the closure of a polyhedron obtained from disjunctions associated with any family of integral lattice-free polyhedra with bounded max-facet width is a rational polyhedron. In the following theorem we use such result. In the remainder of the paper, we call *affine unimodular transformations* the affine transformations that map $\mathbb{Z}^m$ onto $\mathbb{Z}^m$.

Theorem 3 *For any rational polyhedron P, the set $\mathcal{L}(P)$ is again a rational polyhedron.*

Proof Let P be a polyhedron in $\mathbb{R}^{m+n}$. It follows by definition of integral lattice-free cut, that each integral lattice-free cut for P, which is not a valid inequality for P, corresponds to an integral lattice-free polyhedron of $\mathbb{R}^m$ of dimension m. Given two lattice-free polyhedra L, L' in $\mathbb{R}^m$ of dimension m and with $L \subseteq L'$, then clearly $\operatorname{relint} L \subseteq \operatorname{relint} L'$. It follows that each irredundant integral lattice-free cut for P corresponds to an integral lattice-free polyhedron in $\mathbb{R}^m$ of dimension m which is maximal with respect to inclusion. Thus let Z be the family of such maximal lattice-free polyhedra. A result by Averkov et al. [2] implies that the set Z is finite up to affine unimodular transformations.

We now show that for every $L \in Z$, $w(L)$ is finite. By contradiction assume that $w(L)$ is not finite. Thus there exists a facet defining inequality $ax \leq \beta$ of L such that $\beta - \min\{ax : x \in L\}$ is not finite. It follows that there exists a vector $v \in \operatorname{char.cone} L$ with $av < 0$. Let $L' := L + \operatorname{lin.hull}\{v\}$. Since $ax \leq \beta$ is valid for L, $v \notin \operatorname{lin.space} L$, thus $L \subset L'$. Since L is lattice-free and $v \in \operatorname{char.cone} L$, it can be checked that also L' is lattice-free. Moreover since L is integral, also L' is integral. Hence $L' \in Z$ and $L \subset L'$, but this contradicts the maximality of L. Thus $w(L)$ is finite for every $L \in Z$.

It is well-known that affine unimodular transformations preserve the max-facet-width of polyhedra. Since moreover Z is finite up to affine unimodular transformations, and every polyhedron in Z has bounded max-facet-width, it follows that Z is a family of integral lattice-free polyhedra with bounded max-facet-width. It follows by the proof of [1, Theorem 4.3] that the set $\mathcal{L}(P)$ is a rational polyhedron. □

The main result of this paper is that, for each polyhedron P there exists a number k such that $\mathcal{L}^k(P) = P_I$. Our proof of this result is quite technical. It requires two

Lemmas that we state first. In order to streamline the presentation, we postpone the proof of Lemma 10 to Sect. 4. At this point, let us just mention that Lemma 10 applied to polytopes has already been proven by Jörg [8]. We adapt his proof technique to show the general result.

Lemma 10 *Let P be a polyhedron in $\mathbb{R}^{m+n}$, and let $cx+dy \leq \gamma$ be a valid inequality for P_I such that*

$$\operatorname*{proj}_x\{(x, y) \in P_I \;:\; cx + dy = \gamma\}$$

is not lattice-free. Then $\exists k \in \mathbb{N}$ such that $cx + dy \leq \gamma$ is valid for $\mathcal{S}^k(P)$.

Lemma 11 *Let P be an integral lattice-free polyhedron in $\mathbb{R}^m$. Then there exists an integral lattice-free polyhedron $L \supseteq P$ of dimension m such that* relint $P \subseteq$ relint L.

Proof If $\dim P = m$ then $L = P$, so we now assume $d := \dim P < m$. Since the lemma is invariant under affine unimodular transformations, we may assume that $\text{aff.hull}\, P = \mathbb{R}^d \times \{0\}^{m-d}$. It is then easy to verify that $L := P + (\{0\}^d \times \mathbb{R}^{m-d})$ is a lattice-free polyhedron of dimension m with relint $P \subseteq$ relint L. □

We now prove our main result.

Theorem 4 *For each rational polyhedron P there exists $k \in \mathbb{N}$ such that*

$$\mathcal{L}^k(P) = P_I.$$

Proof Let $P \subseteq \mathbb{R}^{m+n}$ be a polyhedron. If $P_I = \emptyset$, then by Theorem 2, $\lim_{i\to\infty} \mathcal{S}^i(P) = \emptyset$, which implies that there exists $k \in \mathbb{N}$ such that $\mathcal{S}^k(P) = \emptyset$. Since $\mathcal{L}^k(P) \subseteq \mathcal{S}^k(P)$, it follows that $\mathcal{L}^k(P) = \emptyset$. Thus, we now assume that $P_I \neq \emptyset$.

Since $P_I \neq \emptyset$, to prove the theorem we show that for every inequality $cx+dy \leq \gamma$ valid for P_I, there exists $k \in \mathbb{N}$ such that $cx + dy \leq \gamma$ is valid for $\mathcal{L}^k(P)$. Note that it suffices to show this because we only need to show it for some finitely many irredundant inequalities defining P_I. One can then take the maximum of the $k \in \mathbb{N}$ for each of these finitely many inequalities. We prove this by induction on $\dim F$, where $F := \{(x, y) \in P_I : cx + dy = \gamma\}$.

We prove the first two base cases. If $F = \emptyset$, then there exists $\epsilon > 0$ such that $cx + dy \leq \gamma - \epsilon$ is valid for P_I. By Theorem 2, $\lim_{i\to\infty} \mathcal{S}^i(P) = P_I$, and by Observation 5, there exists $k \in \mathbb{N}$ such that $cx + dy \leq \gamma - \epsilon + \epsilon = \gamma$ is valid for $\mathcal{S}^k(P)$, and so for $\mathcal{L}^k(P)$. Now let F be a minimal face of P_I. Then F is an affine space and it contains x-integral vectors. It follows that $\text{proj}_x F$ is an affine space too, and it contains integral vectors. Since $\text{proj}_x F$ is an affine space, and the relative interior of every affine space is the same affine space, then $\text{proj}_x F$ contains integral vectors in its relative interior, thus it is not lattice-free. Then by Lemma 10, there exists $k \in \mathbb{N}$ such that $cx + dy \leq \gamma$ is valid for $\mathcal{S}^k(P)$, and so for $\mathcal{L}^k(P)$.

To prove the inductive step, assume that F is a proper face of P_I which is not minimal, and assume that the statement is true for every face of P_I of dimension strictly smaller than $\dim F$. If $\text{proj}_x F$ is not lattice-free, then the statement follows from Lemma 10. Hence, we now assume that $\text{proj}_x F$ is lattice-free.

Since $\text{proj}_x F$ is an integral lattice-free polyhedron, it follows by Lemma 11 there exists an integral lattice-free polyhedron $L \subseteq \mathbb{R}^m$ of dimension m such that $\text{proj}_x F \subseteq L$ and $\text{relint}(\text{proj}_x F) \subseteq \text{relint}\, L$. Now let $a_j x \leq \beta_j$, $j \in J$, be a minimal system of inequalities defining L. The result follows if we show that there exists a $k \in \mathbb{N}$ such that $cx + dy \leq \gamma$ is an integral lattice-free cut for the polyhedron $\mathcal{L}^k(P)$. By definition of integral lattice-free cut, this happens if $cx + dy \leq \gamma$ is valid for every set

$$\{(x, y) \in \mathcal{L}^k(P) \,:\, a_j x \geq \beta_j\}, \quad j \in J.$$

Claim 5 *For every $j \in J$, there exists an inequality $c_j x + dy \leq \gamma_j$ such that:*

(i) *$c_j x + dy \leq \gamma_j$ is valid for P_I;*
(ii) *$F_j := \{(x, y) \in P_I \,:\, c_j x + dy = \gamma_j\} \subset F$;*
(iii) *$cx + dy \leq \gamma$ is valid for $\{(x, y) \in \mathbb{R}^{m+n} \,:\, c_j x + dy \leq \gamma_j,\, a_j x \geq \beta_j\}$.*

Proof of Claim Let $j \in J$. For every $\epsilon \geq 0$, consider the inequality $(\epsilon a_j + c)x + dy \leq \epsilon\beta_j + \gamma$.

(i) Notice that, since $\max\{cx + dy \,:\, (x, y) \in P_I\}$ is attained in the face F, since $a_j x \leq \beta_j$ is valid for F, and since $\lim_{\epsilon \to 0} \epsilon a_j = 0$, it follows that there exists $\overline{\epsilon} > 0$ small enough such that $\max\{(\overline{\epsilon} a_j + c)x + dy \,:\, (x, y) \in P_I\}$ is attained in a face of F. Let $c_j := \overline{\epsilon} a_j + c$, and $\gamma_j := \overline{\epsilon}\beta_j + \gamma$. Since both inequalities $cx + dy \leq \gamma$, and $a_j x \leq \beta_j$ are valid for F, it follows that also their conic combination $c_j x + dy \leq \gamma_j$ is valid for F. Since $\max\{c_j x + dy \,:\, (x, y) \in P_I\}$ is attained in a face of F, and $c_j x + dy \leq \gamma_j$ is valid for F, then $c_j x + dy \leq \gamma_j$ is valid for P_I.
(ii) Since $c_j x + dy \leq \gamma_j$ is valid for P_I, and $\max\{c_j x + dy \,:\, (x, y) \in P_I\}$ is attained in a face of F, then $F_j \subseteq F$. To prove that the inclusion is proper, let $(\overline{x}, \overline{y}) \in F$ with $\overline{x} \in \text{relint}(\text{proj}_x F)$. Since $(\overline{x}, \overline{y}) \in F$, then $c\overline{x} + d\overline{y} \leq \gamma$. Moreover, since $\overline{x} \in \text{relint}(\text{proj}_x F)$, then $\overline{x} \in \text{relint}\, L$, hence $a_j \overline{x} < \beta_j$. Since $\overline{\epsilon} > 0$, it follows that $(\overline{\epsilon} a_j + c)\overline{x} + d\overline{y} < \overline{\epsilon}\beta_j + \gamma$. Hence $(\overline{x}, \overline{y}) \in F$ does not satisfy $c_j x + dy = \gamma_j$, implying $F_j \subset F$.
(iii) Follows by definition of the inequality $c_j x + dy \leq \gamma_j$, and the fact that $\overline{\epsilon} > 0$.

□

For each $j \in J$, let $c_j x + dy \leq \gamma_j$ be an inequality as in Claim 5. We show next that there exists $k \in \mathbb{N}$ such that all the inequalities $c_j x + dy \leq \gamma_j$, $j \in J$, are valid for $\mathcal{L}^k(P)$. Let $j \in J$. By Claim 5(i) $c_j x + dy \leq \gamma_j$ is valid for P_I. It follows by Claim 5(ii) that the set F_j is a face of F different from F, which implies that $\dim F_j < \dim F$. Thus by hypothesis of the induction, there exists $k \in \mathbb{N}$ such that $c_j x + dy \leq \gamma_j$ is valid for $\mathcal{L}^k(P)$. Since J is finite, it follows that there exists $k \in \mathbb{N}$ such that all the inequalities $c_j x + dy \leq \gamma_j$, $j \in J$, are valid for $\mathcal{L}^k(P)$.

Finally by Claim 5(iii), the inequality $cx + dy \leq \gamma$ is valid for every polyhedron $\{(x, y) \in \mathbb{R}^{m+n} \,:\, c_j x + dy \leq \gamma_j,\, a_j x \geq \beta_j\}$, $j \in J$. Since each inequality $c_j x + dy \leq \gamma_j$, $j \in J$, is valid for $\mathcal{L}^k(P)$, it follows that $cx + dy \leq \gamma$ is valid for every polyhedron $\{(x, y) \in \mathcal{L}^k(P) : a_j x \geq \beta_j\}$, $j \in J$. This implies that $cx + dy \leq \gamma$ is an integral lattice-free cut for the polyhedron $\mathcal{L}^k(P)$. $\square$

We conclude this section with an example that shows that generally it is not true that $\mathcal{L}(P) = P_I$. Let $P \subseteq \mathbb{R}^{2+1}$ be the convex hull of the vectors

$$(-1/2, 1/2, 0), (1/2, -1/2, 0), (1/2, 3/2, 0), (3/2, 1/2, 0), (1/2, 1/2, 1).$$

It follows that

$$P_I = \text{conv.hull}(P \cap (\mathbb{Z}^2 \times \mathbb{R})) = \text{conv.hull}\{(0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 1, 0)\}.$$

To show that $\mathcal{L}(P) \neq P_I$ we show that the point $(1/2, 1/2, 1/4) \in P \setminus P_I$ satisfies every integral lattice-free cut for P.

Let $cx + dy \leq \gamma$ be an integral lattice-free cut for P, and let L be an integral lattice-free polyhedron in $\mathbb{R}^2$ corresponding to $cx + dy \leq \gamma$. Clearly we can assume that L is maximal among the integral lattice-free polyhedra in $\mathbb{R}^2$. It follows that L is an affine unimodular transformation of either $S := \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$, or $T := \{x \in \mathbb{R}^2 : x \geq 0, x_1 + x_2 \leq 2\}$.

It can be checked that there exists a point $\overline{x}$ among $(0, 1/2)$, $(1/2, 0)$, $(1/2, 1)$, $(1, 1/2)$ which is not contained in relint L. In fact, if L is an affine unimodular transformation of S this is trivial, while if L is an affine unimodular transformation of T it is easy to see that relint L contains only three vectors x such that $2x \in \mathbb{Z}^2$.

Note that by definition of P, $(\overline{x}, 1/2) \in P$. It follows that $cx + dy \leq \gamma$ is valid for $(\overline{x}, 1/2)$. Since moreover the vectors $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 0)$, $(1, 1, 0)$ are in P_I, the inequality $cx + dy \leq \gamma$ is valid for all of them. It follows that $cx + dy \leq \gamma$ is also valid for conv.hull$\{(\overline{x}, 1/2), (0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 1, 0)\} \ni (1/2, 1/2, 1/4)$. Hence $\mathcal{L}(P) \neq P_I$.

4 Proofs of technical lemmas

Lemma 8 *Let P be a polyhedron in $\mathbb{R}^{m+n}$, let $v \in \mathbb{Q}^{m+n}$ be a nonzero vector, and let H be a subspace that is mixed integer invariant under projection along v. Then*

$$\operatorname*{proj}_{v,H} \mathcal{S}^i(P) \subseteq \mathcal{S}^i(\operatorname*{proj}_{v,H} P) \quad \textit{for every } i \in \mathbb{N}.$$

Proof The proof is by induction on $i \geq 0$, the case $i = 0$ being trivial. We now show the base case $i = 1$, i.e. that $\text{proj}_{v,H} \mathcal{S}(P) \subseteq \mathcal{S}(\text{proj}_{v,H} P)$. Let $\overline{P} := \text{proj}_{v,H} P$, and let $\overline{z} \notin \mathcal{S}(\overline{P})$. We want to show that $\overline{z} \notin \text{proj}_{v,H} \mathcal{S}(P)$. If $\overline{z} \notin \overline{P}$, then clearly $\overline{z} \notin \text{proj}_{v,H} \mathcal{S}(P)$, as $\mathcal{S}(P) \subseteq P$. So we now assume $\overline{z} \in \overline{P}$. Thus there exists a split cut $cz \leq \gamma$ for $\overline{P}$ such that $c\overline{z} > \gamma$. This implies that there exist $a \in \mathbb{Z}^m$, $\beta \in \mathbb{Z}$ such that $cz \leq \gamma$ is valid for both $\overline{P} \cap Q_1$ and $\overline{P} \cap Q_2$, where $Q_1 = \{(x, y) \in \mathbb{R}^{m+n} : ax \leq \beta\}$, $Q_2 = \{(x, y) \in \mathbb{R}^{m+n} : ax \geq \beta + 1\}$.

Now let $c'z \leq \gamma'$ be the inequality defining the half-space

$$\{z \in H \; : \; cz \leq \gamma\} + \text{lin.hull}\{v\}.$$

Notice that by construction $c'v = 0$, and $\{z \in H \; : \; cz \leq \gamma\} = \{z \in H \; : \; c'z \leq \gamma'\}$. Moreover, let $Q'_j := (Q_j \cap H) + \text{lin.hull}\{v\}$, for $j \in \{1, 2\}$, and notice that $Q'_j \cap H = Q_j \cap H$ for $j \in \{1, 2\}$. Since every x-integral vector is contained in $Q_1 \cup Q_2$, it follows that every x-integral vector in H is in $Q'_1 \cup Q'_2$. Since H is mixed integer invariant under projection along v, it follows by definition of Q'_1, Q'_2 that every x-integral vector is contained in $Q'_1 \cup Q'_2$. Note that by construction Q'_1 and Q'_2 are disjoint and defined by two parallel hyperplanes. Moreover $Q'_1 \cup Q'_2 \neq \mathbb{R}^{m+n}$ since $\overline{z} \notin Q'_1 \cup Q'_2$. Hence it can be verified that there exist $a' \in \mathbb{Z}^m$, and $\beta' \in \mathbb{Z}$, such that $Q'_1 \supseteq (Q'_1)_I = \{(x, y) \in \mathbb{R}^{m+n} \; : \; a'x \leq \beta'\}$, and $Q'_2 \supseteq (Q'_2)_I = \{(x, y) \in \mathbb{R}^{m+n} \; : \; a'x \geq \beta' + 1\}$.

Since $cz \leq \gamma$ is valid for $\overline{P} \cap Q_1$, and $\overline{P} \subseteq H$, it follows that $c'z \leq \gamma'$ is valid for $\overline{P} \cap Q_1$, and so for $\overline{P} \cap (Q'_1)_I$. Since $c'v = 0$, it follows that $c'z \leq \gamma'$ is valid for $(\overline{P} \cap (Q'_1)_I) + \text{lin.hull}\{v\} \supseteq P \cap (Q'_1)_I$. Symmetrically $c'z \leq \gamma'$ is valid for $P \cap (Q'_2)_I$. Hence $c'z \leq \gamma'$ is a split cut for P. Since $\overline{z} \in H$, and $c\overline{z} > \gamma$, it follows that $c'\overline{z} > \gamma'$. Since moreover $c'v = 0$, then $\overline{z} \notin \text{proj}_{v,H}\{(x, y) \in \mathbb{R}^{m+n} \; : \; c'z \leq \gamma'\}$. Finally, since $c'z \leq \gamma'$ is a split cut for P, it follows that $\overline{z} \notin \text{proj}_{v,H} \mathcal{S}(P)$. Thus we showed $\text{proj}_{v,H} \mathcal{S}(P) \subseteq \mathcal{S}(\text{proj}_{v,H} P)$.

To prove the inductive step, let $i \geq 2$, and assume that the statement is true for every $0 \leq j \leq i - 1$. Then

$$\begin{aligned}\operatorname*{proj}_{v,H} \mathcal{S}^i(P) &= \operatorname*{proj}_{v,H} \mathcal{S}(\mathcal{S}^{i-1}(P)) \subseteq \mathcal{S}(\operatorname*{proj}_{v,H} \mathcal{S}^{i-1}(P)) \\ &\subseteq \mathcal{S}(\mathcal{S}^{i-1}(\operatorname*{proj}_{v,H} P)) = \mathcal{S}^i(\operatorname*{proj}_{v,H} P).\end{aligned}$$

□

Lemma 9 *Let P be an unbounded polyhedron in $\mathbb{R}^{m+n}$, let v be a nonzero vector in* char.cone P, *and let H be a subspace that is mixed integer invariant under projection along v. Then*

$$\operatorname*{proj}_{v,H}(P_I) = (\operatorname*{proj}_{v,H} P)_I.$$

Proof Let $\overline{P} = \text{proj}_{v,H} P$. Clearly $\overline{P}_I$ is x-integral, and, since P_I is x-integral and H is mixed integer invariant under projection along v, it follows that also $\text{proj}_{v,H}(P_I)$ is x-integral. Thus we only need to show that $\text{proj}_{v,H}(P_I) \cap (\mathbb{Z}^m \times \mathbb{R}^n) = \overline{P}_I \cap (\mathbb{Z}^m \times \mathbb{R}^n)$.

Let $z \in \text{proj}_{v,H}(P_I) \cap (\mathbb{Z}^m \times \mathbb{R}^n)$. Clearly $z \in \overline{P}$, and since z is x-integral, it follows that $z \in \overline{P}_I$.

To prove the converse, let $z \in \overline{P}_I \cap (\mathbb{Z}^m \times \mathbb{R}^n)$. Since in particular $z \in \overline{P}$, it follows there exists a scalar λ such that $z + \lambda v \in P$. As z is x-integral, and v is rational, it follows that there exists a scalar $\mu \geq 0$ such that $w = z + \lambda v + \mu v$ is x-integral. Since

$z + \lambda v \in P$, and v is a nonzero vector in char.cone P, it follows that $w \in P$. This implies $w \in P_I$ and so $z = \text{proj}_{v,H}\, w \in \text{proj}_{v,H}(P_I)$. □

Lemma 10 *Let P be a polyhedron in $\mathbb{R}^{m+n}$, and let $cx + dy \leq \gamma$ be a valid inequality for P_I such that*

$$\operatorname*{proj}_x \{(x, y) \in P_I \,:\, cx + dy = \gamma\}$$

is not lattice-free. Then $\exists k \in \mathbb{N}$ such that $cx + dy \leq \gamma$ is valid for $\mathcal{S}^k(P)$.

Proof We define the sets $F := \{(x, y) \in P_I \,:\, cx + dy = \gamma\}$ and $Q := \{(x, y) \in P \,:\, cx + dy > \gamma\}$. If $Q = \emptyset$ there is nothing to show, thus we assume $Q \neq \emptyset$. Let $\overline{Q} := \{(x, y) \in P \,:\, cx + dy \geq \gamma\}$ be the topological closure of Q. Since $cx + dy \leq \gamma$ is valid for P_I, then Q does not contain any x-integral vector. It follows that $\text{proj}_x Q$ does not contain any integral vector, and that $\text{proj}_x \overline{Q}$ is lattice-free, since it is the topological closure of $\text{proj}_x Q$.

Now we show that there exists an inequality $ax \geq \beta$, with $a \in \mathbb{Z}^m, \beta \in \mathbb{Z}$, such that $ax = \beta$ is valid for F, and $ax > \beta$ is valid for Q. $F \subseteq \overline{Q}$ implies that $\text{proj}_x F \subseteq \text{proj}_x \overline{Q}$, and since $\text{proj}_x \overline{Q}$ is lattice-free while by hypothesis $\text{proj}_x F$ is not, $\dim(\text{proj}_x F) < \dim(\text{proj}_x \overline{Q})$ and $\text{proj}_x F$ is contained in a proper face of $\text{proj}_x \overline{Q}$. Let G be a minimal face of $\text{proj}_x \overline{Q}$, with respect to inclusion, containing $\text{proj}_x F$. It follows by the minimality assumption that G is not lattice-free. Now let $ax \geq \beta$ be valid for $\text{proj}_x \overline{Q}$ and such that $G = \{x \in \text{proj}_x \overline{Q} \,:\, ax = \beta\}$. Then clearly $ax = \beta$ is valid for $\text{proj}_x F$ and so for F. Moreover, since $\text{proj}_x Q$ contains no integral point, and since G is not lattice-free, then it is easy to verify that $ax > \beta$ is valid for $\text{proj}_x Q$ and so for Q. Clearly, since a is rational and $\text{proj}_x F$ contains integral vectors, we can assume that $a \in \mathbb{Z}^m$ and $\beta \in \mathbb{Z}$.

We now complete the proof by showing that there exists $k \in \mathbb{N}$ such that $cx + dy \leq \gamma$ is a split cut for $\mathcal{S}^k(P)$. We introduce the sets $Q^i := \{(x, y) \in \mathcal{S}^i(P) \,:\, cx + dy > \gamma\}$ and $\overline{Q}^i := \{(x, y) \in \mathcal{S}^i(P) \,:\, cx + dy \geq \gamma\}$ for every $i \in \mathbb{N}$. By Theorem 2, $\lim_{i\to\infty} \mathcal{S}^i(P) = P_I$. By intersecting P_I and $\mathcal{S}^i(P)$ for every $i \in \mathbb{N}$ with the half-space corresponding to the inequality $cx + dy \geq \gamma$, it follows by Observation 6 that $\lim_{i\to\infty} \overline{Q}^i = F$. Since $ax = \beta$ is valid for F, it follows by Observation 5 that there exists $k \in \mathbb{N}$ such that $ax < \beta + 1$ is valid for $\overline{Q}^k$. Since $Q^i \subseteq \overline{Q}^i$ for every $i \in \mathbb{N}$, it follows that $ax < \beta + 1$ is valid for Q^k. Moreover, since $Q^i \subseteq Q$ for every $i \in \mathbb{N}$, and since $ax > \beta$ is valid for Q, it follows that $ax > \beta$ is valid for Q^k. Hence $\beta < ax < \beta + 1$ is valid for Q^k. In other words, $cx + dy \leq \gamma$ is valid for both

$$\{(x, y) \in \mathcal{S}^k(P) \,:\, ax \leq \beta\} \quad \text{and} \quad \{(x, y) \in \mathcal{S}^k(P) \,:\, ax \geq \beta + 1\},$$

implying that $cx + dy \leq \gamma$ is a split cut for $\mathcal{S}^k(P)$. □

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References

1. Andersen, K., Louveaux, Q., Weismantel, R.: An analysis of mixed integer linear sets based on lattice point free convex sets. Math. Oper. Res. **35**, 233–256 (2010)
2. Averkov, G., Wagner, C., Weismantel, R.: Maximal lattice-free polyhedra: finiteness and an explicit description in dimension three. Math. Oper. Res. http://arxiv.org/abs/1010.1077 (2011, to appear)
3. Blaschke, W.: Kreis und Kugel. Chelsea, New York (1949)
4. Chvátal, V.: Edmonds polytopes and a hierarchy of combinatorial problems. Discret. Math. **4**, 305–337 (1973)
5. Cook, W.J., Kannan, R., Schrijver, A.: Chvátal closures for mixed integer programming problems. Math. Program. **47**, 155–174 (1990)
6. Gomory, R.E.: An algorithm for integer solutions to linear programs. In: Graves, R.L., Wolfe, P. (eds.) Recent Advances in Mathematical Programming, pp. 269–302. McGraw-Hill, New York (1963)
7. Gruber, P.M., Lekkerkerker, C.G.: Geometry of Numbers. 2nd edn. North-Holland Mathematical Library, Amsterdam (1987)
8. Jörg, M.: k-disjunctive cuts and cutting plane algorithms for general mixed integer linear programs. Ph.D. Thesis, Technische Universität München, München (2008)
9. Meyer, R.R.: On the existence of optimal solutions to integer and mixed-integer programming problems. Math. Program. **7**, 223–235 (1974)
10. Owen, J.H., Mehrotra, S.: A disjunctive cutting plane procedure for general mixed integer linear programs. Math. Program. **89**, 437–448 (2001)
11. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (1970)
12. Salinetti, G., Wets, R.J.-B.: On the convergence of sequences of convex sets in finite dimensions. Soc. Ind. Appl. Math. **21**, 18–33 (1979)
13. Schrijver, A.: On cutting planes. Ann. Discret. Math. **9**, 291–296 (1980)
14. Schrijver, A.: Theory of Linear and Integer Programming. Wiley, Chichester (1986)